Introduction and Vagueness

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Philosophical Logic 2025
27 October 2025

Course Team

- Lecturer: Marco Degano m.degano@uva.nl
- 2 lectures per week: Monday, Wednesday, 15-17, SP D1.113
- TA: Tomasz Klochowicz t.j.klochowicz@uva.nl
- 1 tutorial per week: Thursday (varying times)

https://datanose.nl/#course[137525]

Course Aim

- **Technical toolkit:** The course aims to equip you with a basic toolbox for formally representing and evaluating philosophical problems and arguments, which can be relevant for later (MoL) courses and projects.
- **Content:** We will study a variety of non-classical logics, and we will critically assess how well these systems capture the phenomena they are designed to model.
- Philosophical discussion will motivate the topics, but it will remain focused, as this course is not a seminar-style deep dive on a single theme.

Materials

- Primary materials: lecture slides. Slides will be available on Canvas before each lecture.
- On Canvas, there are also lecture notes by Levin Hornischer covering core topics of the *Philosophical Logic* course (and beyond), but the presentation might differ from the lectures.
- Suggested and further readings provided at the start of each slide set.

Schedule

Date	Time	Room	Topic	Notes
27 Oct	15-17	SP D1.113	Introduction, Paradoxes & Vagueness	
29 Oct	15-17	SP D1.113	Many-valued Logics	Assignment 1 out
30 Oct	15-17	SP F1.02	Tutorial	
3 Nov	15-17	SP D1.113	Supervaluationism	
5 Nov	15-17	SP D1.113	Vagueness: Other Approaches	Assignment 2 out
6 Nov	13-15	SP L0.12	Tutorial	
10 Nov	15-17	SP D1.113	Kripke's Theory of Truth	
12 Nov	15-17	SP D1.113	Guest Lecture Robert van Rooij on Self-referential truth paradoxes	Assignment 3 out
13 Nov	13-15	SP A1.04	Tutorial	

Schedule

Date	Time	Room	Topic	Notes
17 Nov	15-17	SP D1.113	Truthmaker Semantics	
19 Nov	15-17	SP D1.113	Guest Lecture Robert van Rooij on Relevance, FDE and Truth-making	Assignment 4 out
20 Nov	13-15	SP A1.04	Tutorial	
24 Nov	15-17	SP D1.113	Counterfactuals	
26 Nov	15-17	SP D1.113	Counterfactuals	Assignment 5 out
27 Nov	15-17	SP F1.02	Tutorial	
1 Dec	15-17	SP D1.113	Non-monotonic Logics	
3 Dec	15-17	SP D1.113	Logic, Probability and Conditionals	Practice exam out
4 Dec	13-15	SP C1.112	Tutorial	
8 Dec	15-17	SP D1.113	Logic, Probability and Conditionals	
10 Dec	15-17	SP D1.113	Paradoxes and Inclosure Schema	
11 Dec	13-15	SP F2.04	Tutorial	
18 Dec	9-11	SP F2.04	Final Exam	

Assessment

- **Overall grade:** 50% take-home assignments + 50% final exam.
- **Assignments:**
  - Each set combines a **philosophical exercise** (short writing/reflection) and **technical exercises**.
  - Released every Tuesday at 12:00; due the following Tuesday at 21:00.
  - There will be 5 assignments in total. The lowest score is dropped.
  - Discussion with classmates is encouraged, but all submitted work must be written **individually**.
- The exam will include technical exercises as well as one philosophical question.

Course Feedback

- The university will run a detailed evaluation at the end of the course, which is valuable for long-term improvements.
- For ongoing improvements, we welcome feedback **during** the course: suggestions, concerns, or criticism.
- You can always contact us directly, or (if you prefer to remain anonymous) use the feedback form:

`https://forms.gle/mnsww6H7se3rn4qx9`

Plan

1. Philosophical Logic and Paradoxes

2. Vagueness

3. Three-valued Logics

Readings

Suggested:

- Lecture notes: ch. 1; ch. 2.1-2.2; ch. 3.1-3.2.1

Further reading:

- An Introduction to Non-Classical Logic (Priest): ch. 1.1-1.3; ch. 7.1-7.3
- Logic for Philosophy (Sider): ch. 3.4.1-3.4.3
- Philosophical Logic (MacFarlane): ch. 8.1-8.2

Outline

Philosophical Logic vs Philosophy of Logic

- What is the role played by 'logic' in philosophical *logic*?
- **Logic:** Formal system to regiment *reasoning* by means of a formal language (e.g., rules of inference, valid inference, completeness, consistency, axiomatization, . . .).
- **Philosophy of Logic:** the philosophical study of 'logic' and its fundamental concepts (e.g., the nature of *entailment*, the *truth* of a logical statement, . . .)
- **Philosophical Logic:** the application of logic(s) to philosophical problems (e.g., knowledge and epistemic logics, conditionals, vagueness, . . .)
- The domains of inquiry of philosophy of logic and philosophical logic are in many respects interconnected, and this course engages with both.

Philosophical logic and classical logic

- Another way to conceive philosophical logic is the study of logics which are non-classical (intuitionistic logic, relevance logic, . . .).
- The law of excluded middle is valid in classical logic. But it is not valid in intuitionistic logic.

$$p \vee \neg p$$

- Intuitionistic logic rejects non-constructive proofs and links 'truth' with 'verifiability'.

[Burgess 2009; Prawitz 2005]

Example: classical vs. constructive

There exist irrational x, y such that x^y is rational.

Let $\alpha := \sqrt{2}^{\sqrt{2}}$. By excluded middle, either $\alpha \in \mathbb{Q}$ or $\alpha \notin \mathbb{Q}$.

- If $\alpha \in \mathbb{Q}$, take $x = y = \sqrt{2}$. Then $x^y = \alpha \in \mathbb{Q}$.
- If $\alpha \notin \mathbb{Q}$, take $x = \alpha$ and $y = \sqrt{2}$. Then $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$.

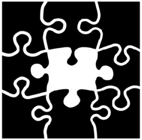
- A constructive proof of $A \vee B$ must exhibit a proof of *either* A *or* B, with an indication of which.
- The argument above appeals to excluded middle without identifying which disjunct holds and hence it is non-constructive.
- (In fact $\sqrt{2}^{\sqrt{2}}$ is *irrational* (indeed transcendental). A proof of its irrationality exists but is nontrivial.)

Intuitionistic logic and this course

- Intuitionistic logic is an important topic in philosophical logic.
- Being studied and developed in Amsterdam, it is well represented in the MoL:
  - Philosophical aspects: *Philosophy of mathematics*
  - Proof-theoretic aspects: *Proof theory*
  - Algebraic aspects: *Mathematical structures in logic*
- We will not cover intuitionistic logic here, and focus on other core topics important for your training in philosophical logic.

Philosophical logic and paradoxes

Many philosophically interesting problems and logics emerge as solutions seeking to solve a particular **paradox** or **puzzle**.



Paradoxes will often be our starting point, leading to the study of logical theories which can account for them.

Omnipotence and the paradox of the stone

Could God create a stone so heavy that even God could not lift it?

1. God is omnipotent (i.e., God can do anything)
2. If God can create a stone that God *can not* lift, then God is not omnipotent.
3. If God *can not* create a stone that God can not lift, then God is not omnipotent.
4. God is not omnipotent.

The Liar Paradox

A: This sentence is false

- *Assume* A *true.* Then it is false. So if it's true, it's false.
- *Assume* A *is false.* Then it is true. So if it's false, it's true.

Epistemology and self-refutation

- *Relativism:* There is no absolute truth.
- *Scepticism:* Nothing can be known.

- Do relativists take as **true** that there is no absolute **truth**?
- Do sceptics **know** that nothing can be **known**?

Russell's Paradox

$$A := \{X \mid X \notin X\}$$

- A is the set of all sets that are not members of themselves.
- Assume $A \in A$. Then, by the definition of A, it must be that $A \notin A$. Contradiction.
- Assume $A \notin A$. Then, by the definition of A, it must be that $A \in A$. Contradiction.

Berry's Paradox

$\boxed{\textit{The smallest positive integer not definable in under 200 letters.}}$

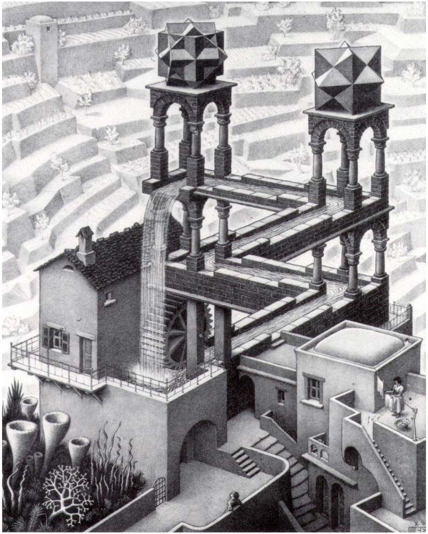
- Let X be the set of positive integers definable in under 200 letters (in English). Since there are only finitely many sub-200-letter sentences, X is finite. So there are numbers *not* in X. By leastness, one of them is smallest.
- The boxed phrase appears to *define* exactly that smallest number, and it does so using fewer than 200 letters, which says, of that very number, that it is *not* definable under 200 letters.

The role of paradoxes

- All these paradoxes rest on a number of (controversial) assumptions.
- We will explore how to identify the relevant assumptions associated with a paradox and determine its *structure*.
- By doing so, we will be able to show that different paradoxes exhibit similar structures and thus call for a unitary solution.

Paradoxes

But what is a paradox?



Waterfall - M. C. Escher

Defining a paradox

A paradox is an argument with **assumptions** that appear true and steps that appear valid, yet it yields an unacceptable (e.g., contradictory or absurd) conclusion.

An example: the liar paradox

- **Truth predicate** $T(\cdot)$
  - We add T to the *object language* so sentences can talk about the truth of (names of) sentences.
  - *Why:* Without a truth predicate, the language cannot say of itself that something is true/false.
- **Names/quotation** 'A'
  - For any sentence A, we use *'A'* as a *name* for A itself.
  - *Why:* T needs an argument; names let us refer to a sentence *as an object* inside the language.
- T**-schema (transparency)**
  - $T(\text{`}A\text{'}) \;\leftrightarrow\; A$ for every sentence A.
  - *Why:* This links talk of truth to what the sentence says. It licenses replacing $T(\text{`}A\text{'})$ by A and vice versa.

The Liar Paradox

Let A be the sentence *'A' is false*. $A \equiv \neg T(\text{'}A\text{'})$

1. $T(\text{'}A\text{'}) \vee \neg T(\text{'}A\text{'})$ LEM

Case 1: Assume $T(\text{'}A\text{'})$

2. $T(\text{'}A\text{'})$ Assumption
3. A T-schema (from 2)
4. $\neg T(\text{'}A\text{'})$ from 3 and $(A \models \neg T(\text{'}A\text{'}))$
5. $T(\text{'}A\text{'}) \wedge \neg T(\text{'}A\text{'})$ $\wedge$-Intro (2, 4)

Case 2: Assume $\neg T(\text{'}A\text{'})$

6. $\neg T(\text{'}A\text{'})$ Assumption
7. A from 6 and $(\neg T(\text{'}A\text{'}) \models A)$
8. $T(\text{'}A\text{'})$ T-schema (from 7)
9. $T(\text{'}A\text{'}) \wedge \neg T(\text{'}A\text{'})$ $\wedge$-Intro (6, 8)

10. $T(\text{'}A\text{'}) \wedge \neg T(\text{'}A\text{'})$ $\vee$-Elim on 1 with Cases 1–2

An example: the liar paradox

1. $T(`A') \vee \neg T(`A')$ LEM

Case 1: Assume $T(`A')$

2. $T(`A')$ Assumption
3. A T-schema (from 2)
4. $\neg T(`A')$ from 3 and $(A \models \neg T(`A'))$
5. $T(`A') \wedge \neg T(`A')$ $\wedge$-Intro (2, 4)

Case 2: Assume $\neg T(`A')$

6. $\neg T(`A')$ Assumption
7. A from 6 and $(\neg T(`A') \models A)$
8. $T(`A')$ T-schema (from 7)
9. $T(`A') \wedge \neg T(`A')$ $\wedge$-Intro (6, 8)

10. $T(`A') \wedge \neg T(`A')$ $\vee$-Elim on 1 with Cases 1–2

What to give up?

Extra-logical assumptions: truth predicate; T-schema (full $\leftrightarrow$ or one direction).

Logical assumptions: LEM, explosion, $\wedge$-Introduction, reasoning by cases, . . .

The road to philosophical logic

- To challenge a core assumption, you need:
  1. **Philosophical motivation**: clarify why to doubt it, what phenomena to capture, and the costs/benefits.
  2. **Formal discipline**: design a well-structured logic (syntax, semantics, proof theory, . . .) and establish metatheory (soundness, completeness, . . .).
- Bringing these together constitutes the field of philosophical logic.

Outline

Red, tall, and bald

A term is **vague** when its correct application admits *indeterminate/borderline cases* and obeys a *tolerance* idea (small changes shouldn't flip the verdict). So there is no sharp cutoff.

- *Red*: along a hue continuum there is a range where it's unclear whether a patch is red.

- *Tall*: there is a grey zone where neither "tall" nor "not tall" seems clearly correct.
- *Bald*: there is no exact hair count at which someone becomes bald; adding or removing one hair shouldn't change the judgment.

Distinguishing vagueness: Ambiguity

Is vagueness ambiguity?

There is a duck by the BANK.

bank = financial institution *or* riverbank.

- Ambiguity = one expression with *multiple conventional meanings*.
- Disambiguation (by context or paraphrase) selects **a single meaning**.
- **No borderline** cases are required for ambiguity.

Distinguishing vagueness: Context-dependence

Does vagueness = context-dependence?

Not necessarily. Some terms are context-dependent but *crisp*, others remain *vague* even after context is fixed.

- Context-dependent but crisp (not vague): *I*, *today*, *here*.
- Vague & context-dependent: *tall* (threshold varies by group/standard, yet a grey zone remains).
- Vague but not allegedly context-dependent: *bald*, *heap*, *bush*.

Distinguishing vagueness: Underspecification

Is vagueness just underspecification?

Underspecification = missing detail where a precise value still exists.

- Underspecified (not vague): “The meeting is *sometime this afternoon*.” (An exact time exists but isn’t given.)
- Vague: *red*, *heap*, *bald*: small changes shouldn’t flip the verdict (tolerance), so there are indeterminate cases.

Underspecification is about *information left open*. Vagueness is about *tolerance* in meaning.

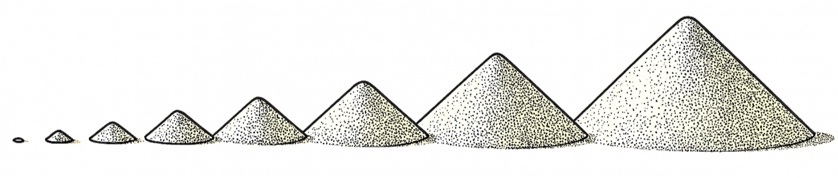
Criteria for Vagueness

- Lack of sharp boundaries
- Presence of indeterminate cases
- Tolerant to small differences along a relevant dimension
- Can lead to the Sorites paradox

The Sorites paradox

1 million grains make a heap.
If 1M grains make a heap, then (1M-1) grains make a heap.
If (1M-1) grains make a heap, then (1M-2) grains make a heap.
...
If 2 grains make a heap, then 1 grain makes a heap.

Therefore, 1 grain makes a heap.



Abstract sorites schema

Many-premise (series; downward from a large N, e.g. 1M):

$$
\begin{array}{l}
F(N) \\
F(N) \rightarrow F(N-1) \\
F(N-1) \rightarrow F(N-2) \\
\vdots \\
F(2) \rightarrow F(1) \\
\hline
\therefore\ F(1)
\end{array}
$$

Inductive (quantified; downward):

$$
\begin{array}{l}
F(N) \\
\forall n\,(F(n) \rightarrow F(n-1)) \\
\hline
\therefore\ \forall k\,(1 \leq k \leq N \Rightarrow F(k)) \text{ (in particular } F(1)\text{).}
\end{array}
$$

Many-premise vs. inductive

- *Premises:*
  - **Many-premise:** $N-1$ separate tolerance instances ($F(n)\to F(n-1)$ for $2 \le n \le N$)
  - **Inductive:** one *unrestricted* universal premise $\forall n\,(F(n)\to F(n-1))$
- *Conclusion:*
  - **Many-premise:** endpoint only (e.g. $F(1)$).
  - **Inductive:** range universal $\forall k\,(1\le k\le N \Rightarrow F(k))$ (hence $F(1)$).
- *Plausibility profile:* Local steps feel compelling, while a *single universal* tolerance claim is easier to doubt.
- *Direction:* Upward vs. downward are just re-indexings. The contrasts above seem invariant.

The Sorites paradox

$$
\begin{array}{l}
F(N) \\
F(N) \to F(N-1) \\
F(N-1) \to F(N-2) \\
\vdots \\
F(2) \to F(1) \\
\hline
\therefore\ F(1)
\end{array}
$$

We have many plausible premises (local tolerance steps) and one rule: **Modus Ponens**. What to give up?

- **Deny a tolerance instance (sharp cutoff)**

$$\exists n \left(F(n) \ \wedge\ \neg F(n-1)\right)$$

- **Non-classical/semantic replies** At some n, $F(n)$ is not simply true/false; MP can't carry the chain through the indeterminate region.
- **Reject Modus Ponens** (rare, most keep MP and revise tolerance/truth conditions).

Why look at three-valued logics?

- In the sorites chain we rely on many "tiny" tolerance steps. At some point, the claim $F(n) \to F(n-1)$ crosses an indeterminate case.
- With only two truth values, each step must be simply *true* or *false*. If we keep all steps true, the chain runs through to $F(1)$. If we deny a step, we've drawn a sharp (and seemingly arbitrary) cutoff.
- **Three-valued idea:**
  - Add a third status for *indeterminate* cases (neither clearly true nor clearly false).
  - Indeterminate steps are *not* treated as plain truths, so the chain cannot keep advancing.

Outline

Classical Logic

- A logic is a formal language with a deductive system and/or a semantics.
- Formal language (object language): set of well-formed strings over a finite alphabet

$$\varphi ::= p|\bot|\top|\neg\varphi|\varphi \vee \varphi|\varphi \wedge \varphi|\varphi \to \varphi$$

- An argument is *derivable* if there is a deductive procedure to derive its conclusion from its premises. An argument is *valid* if whenever its premises are all true, its conclusion is true.

[Priest 2008, ch. 1]

Semantics

Let P be the set of propositional letters. A (classical) **valuation** is a map $v : P \to \{0, 1\}$ that extends to all formulas by recursion:

$$
\begin{aligned}
&v(\top) = 1 \\
&v(\bot) = 0 \\
&v(\neg\varphi) = 1 - v(\varphi) \\
&v(\varphi \wedge \psi) = \min\big(v(\varphi),\, v(\psi)\big) \\
&v(\varphi \vee \psi) = \max\big(v(\varphi),\, v(\psi)\big) \\
&v(\varphi \to \psi) = \max\big(1 - v(\varphi),\, v(\psi)\big) \ \text{ (equivalently, } v(\neg\varphi \vee \psi))
\end{aligned}
$$

The valuation function is often conveniently represented by means of truth-tables.

Some important notions

- A formula φ is **satisfiable** by a valuation v iff $v(\varphi) = 1$
- A formula φ is **valid** iff $v(\varphi) = 1$ for any valuation v.
- Given a set of formulas Γ and a formula φ, we say that Γ **entails** φ (written $\Gamma \models \varphi$) iff for every valuation v, whenever $v(\gamma) = 1$ for all $\gamma \in \Gamma$, it follows that $v(\varphi) = 1$.
- What changes to the logic can be made to depart from the classical picture?

Bivalence and Vagueness

- **Principle of Bivalence:** Every sentence has exactly one of two truth values:

$$T = \{0, 1\} \quad \text{(false / true)}$$

- **Vagueness challenge:** In indeterminate cases (e.g. sorites steps $F(n) \to F(n-1)$) a statement seems neither clearly true nor clearly false.
- **Three-valued idea:** Introduce an additional value for "indeterminate."

$$T = \{0,\, i,\, 1\}$$

Some important notions (in three-valued logics)

- We work with a set of truth values T and a set of *designated* values $T^+ \subseteq T$ (those that count as "true")
- A formula φ is **satisfiable** by a valuation v iff $v(\varphi) \in T^+$
- A formula φ is **valid** iff $v(\varphi) \in T^+$ for every valuation v
- Given a set of formulas Γ and a formula φ, we say that Γ **entails** φ (written $\Gamma \models \varphi$) iff for every valuation v, whenever $v(\gamma) \in T^+$ for all $\gamma \in \Gamma$, it follows that $v(\varphi) \in T^+$

Three-valued logics: what can vary?

- Fix the truth values $T = \{0, i, 1\}$ and a designated set $T^+ \subseteq T$.
- Different *logics* arise by changing:
  - **How the connectives are defined** (their truth tables; how they treat i).
  - **Which values are designated for entailment/validity** (e.g., $T^+ = \{1\}$ vs. $T^+ = \{1, i\}$).
- For the moment, we keep $T^+ = \{1\}$ and focus on logics defined by different connectives.

Truth-tables

$\wedge$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	0	$?$	0

$\vee$	1	i	0
1	1	$?$	1
i	$?$	$?$	$?$
0	1	$?$	0

$\rightarrow$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	1	$?$	1

$\neg$	
1	0
i	$?$
0	1

Take $\wedge$. How many truth value functions can you define? $3^5 = 243$

Some natural constraints:

Idempotence: $p \wedge p \equiv p$

Symmetry: $p \wedge q \equiv q \wedge p$

How many truth value functions for $\wedge$? $3^2 = 9$

Strong Kleene K_3^s

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\to$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

- **Role of i:** indeterminate, unknown
- **Unknown only when needed:** return i *only* if the classical truth-conditions don't already fix the value.
- $i \wedge 1 = i$ (conjunction equals the "weakest link"; not enough info to make it 0 or 1).
- $i \vee 1 = 1$; $i \vee 0 = i$ (a true disjunct suffices; otherwise still undetermined).
- $\neg i = i$ (negation doesn't resolve indeterminacy).

Classicality preservation (Strong Kleene)

Lemma (Classicality preservation)

Let $\mathrm{Var}(\varphi)$ be the set of variables in φ. For every formula φ and valuation v, if $v(p) \in \{0, 1\}$ for every $p \in \mathrm{Var}(\varphi)$, then $v(\varphi) \in \{0, 1\}$.

Proof. By structural induction on φ.

Base (atom). If φ is a variable p, then $v(\varphi) = v(p) \in \{0, 1\}$.

Inductive steps.

$(\neg)$ $\varphi = \neg\alpha$. If v is classical on $\mathrm{Var}(\varphi) \supseteq \mathrm{Var}(\alpha)$, then by IH $v(\alpha) \in \{0, 1\}$. Strong Kleene negation agrees with classical negation on $\{0, 1\}$: $\neg 1 = 0$, $\neg 0 = 1$. Hence $v(\varphi) = v(\neg\alpha) \in \{0, 1\}$.

Classicality preservation (Strong Kleene)

($\wedge$) $\varphi = \alpha \wedge \beta$. If v is classical on $\mathrm{Var}(\alpha) \cup \mathrm{Var}(\beta)$, IH gives $v(\alpha), v(\beta) \in \{0, 1\}$. Strong Kleene $\wedge$ restricted to $\{0, 1\}$ is classical:

$$1 \wedge 1 = 1 \quad 1 \wedge 0 = 0 \quad 0 \wedge 1 = 0 \quad 0 \wedge 0 = 0$$

Thus $v(\varphi) = v(\alpha \wedge \beta) \in \{0, 1\}$.

($\vee, \to$) (omitted, similar.)

All constructors preserve classicality on classical inputs. $\square$

Another look at the valuation function

We can view the undefined value i as $\frac{1}{2}$

$$0 < \frac{1}{2} < 1$$

Then K_3^s admits a clean representation mirroring the classical clauses:

$$\begin{aligned} p \wedge q &= \min(p, q) & \qquad p \vee q &= \max(p, q) \\ \neg p &= 1 - p & \qquad p \to q &= \max(1 - p,\ q) \end{aligned}$$

Thus, K_3^s conservatively extends classical logic. And in fact, it is the most conservative extension: it says $\frac{1}{2}$ exactly when the classical completions disagree.

Weak Kleene K_3^w

$\wedge$	1	i	0
1	1	i	0
i	i	i	i
0	0	i	0

$\vee$	1	i	0
1	1	i	1
i	i	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	i	i	i
0	1	i	1

$\neg$	
1	0
i	i
0	1

- Role of i: *meaninglessness*/*undefinedness*
- Examples: "The present king of France is bald", terms without denotation, *NaN*.
- Intuition: if a component is meaningless, any larger context talking *about it* becomes meaningless.

- *Infectiousness:* for $\star \in \{\wedge, \vee, \rightarrow\}$, $i \star x = x \star i = i$ and $\neg i = i$.

Some Facts

- Both K_3^s and K_3^w have no tautologies. Why?

$$v_\vee(i,i) = v_\wedge(i,i) = v_\rightarrow(i,i) = v_\neg(i) = i$$

- Still, the consequence relation is not trivial. Can you think of some cases where K_3^s and K_3^w diverge?
  $p \models_{K_3^s} p \vee q$ $p \not\models_{K_3^w} p \vee q$

Jan Łukasiewicz (1878-1956)

- Polish logician and philosopher, a leading figure of the Lwów-Warsaw school. Later professor at University College Dublin.
- Pioneered many-valued logic and introduced Polish (prefix) notation.
- He was motivated by Aristotle's problem of *future contingents* (e.g., "There will be a sea battle tomorrow").
- Such future-tensed statements seem neither true nor false *now*.
- Łukasiewicz proposed a third value for the present status of such claims.

Jan Łukasiewicz, 1878-1956

Łukasiewicz three-valued logic Ł3

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	1	$\mathbf{1}$	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

- Conjunction and disjunction behave like in K_3^s.
- The key difference is implication: $i \rightarrow i = 1$.
- Motivation: in classical logic $p \rightarrow p$ is always valid. Łukasiewicz wanted to preserve this tautology even with a third value.
- But note: $p \rightarrow q \not\equiv \neg p \vee q$.

Implication in (three-)valued logics

$$(K_3^s): \quad x \to y = \max(1-x,\, y) = \begin{cases} y & \text{if } y \geq 1-x, \\ 1-x & \text{if } y < 1-x\ , \end{cases}$$

$$(\text{Ł}3): \quad x \to y = \min(1,\, 1-x+y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1-x+y & \text{if } x > y\ , \end{cases}$$

$$(\star): \quad x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y\ . \end{cases}$$

- **Strong Kleene**: treats $\to$ as classical material implication with partial information. The value is the *stronger of* $1-x$ (the cost of the antecedent) and y (the support for the consequent).
- **Łukasiewicz**: reads $x \to y$ as "y is at least as true as x up to compensation". If $x \leq y$ then the conditional is fully true. Otherwise you get a linear downgrade $1-x+y$. (e.g., $\frac{1}{2} \to 0 = \frac{1}{2}$)
- $\star$: no compensation. The conditional is 1 exactly when the conclusion is at least as true as the premise. Otherwise it is just as true as y. (e.g., $\frac{1}{2} \to 0 = 0$)

Set of Tautologies

Let $\mathcal{T}_{\mathrm{CL}}$ and $\mathcal{T}_{Ł3}$ be the sets of tautologies in classical logic and in Ł3. We have that:

$$\mathcal{T}_{Ł3} \subsetneq \mathcal{T}_{CL}$$

Call a valuation v classical if $v(p) \in \{1, 0\}$ for all propositional letters in the language. Show by induction that for any φ, if v is classical, then $v_{Ł3}(\varphi) = v_{CL}(\varphi)$

Functional completeness: what it is and why it matters

Definition (Functional completeness)

Fix a finite nonempty set of truth values T. We say that a set of connectives $\mathcal{C}$ is *functionally complete over* T iff for every $n \geq 0$ and every function $f : T^n \to T$ there exists a $\mathcal{C}$-formula $\varphi_f(p_1, \ldots, p_n)$ such that for all valuations v,

$$v\big(\varphi_f(p_1, \ldots, p_n)\big) \;=\; f\big(v(p_1), \ldots, v(p_n)\big)$$

- *Expressive adequacy:* any finite truth-table is definable.
- *Normal forms:* constructive translation from a table to a formula.

The setting

- Truth values: $T = \{0, \frac{1}{2}, 1\}$.
- Primitives: negation $\neg$ and implication $\to$ with Łukasiewicz semantics

$$\neg x = 1 - x, \qquad x \to y = \min(1,\, 1 - x + y)$$

- Ł$3_{\{\neg,\to\}}$ is clearly not functionally complete.
- To reach **functional completeness**, we **add** the truth-constant $1/2$ to the language.

Theorem (Functional completeness for Ł$3_{\{\neg,\to,1/2\}}$)

Let $T = \{0, \frac{1}{2}, 1\}$ and interpret the connectives by the Łukasiewicz truth-functions:

$$\neg x = 1 - x \qquad x \to y = \min(1,\, 1 - x + y) \qquad 1/2 = \frac{1}{2}$$

Then $\{\neg, \to, 1/2\}$ is functionally complete over T in the sense of the Definition before.

Derived connectives and constants

We use the following derived connectives:

$$x \varovee y \; := \; \neg x \to y \qquad \text{(strong disjunction)}$$
$$x \varowedge y \; := \; \neg(x \to \neg y) \qquad \text{(strong conjunction)}$$

and the classical constants

$$\mathbf{1} := p \to p, \qquad \mathbf{0} := \neg \mathbf{1}$$

Facts (easy checks from the semantics):

- $x \varovee y = \min(1, x + y)$ (accumulates truth by capped addition)
- $x \varowedge y = \max(0, x + y - 1)$ (returns the excess over 1)
- $\frac{1}{2} \varovee \frac{1}{2} = 1$ but $\frac{1}{2} \vee \frac{1}{2} = \frac{1}{2}$; $\frac{1}{2} \varowedge \frac{1}{2} = 0$ but $\frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}$
- if $x, y \in \{0, 1\}$ then $\varovee = \vee$ and $\varowedge = \wedge$

Unary selectors (recognizing a single truth value)

Our goal is to build $\delta_a(x) \in \{0, 1\}$ with $\delta_a(x) = 1$ iff $x = a$.

Define equivalence $x \leftrightarrow a := (x \to a) \oslash (a \to x)$

$$x \leftrightarrow a = \begin{cases} 1 & \text{if } x = a, \\ \frac{1}{2} & \text{if } \{x, a\} = \{0, \frac{1}{2}\} \text{ or } \{\frac{1}{2}, 1\}, \\ 0 & \text{if } \{x, a\} = \{0, 1\}. \end{cases}$$

Let $s(u) := u \oslash u = \max(0, 2u - 1)$. Then $s(1) = 1$ while $s(\frac{1}{2}) = s(0) = 0$.

$$\delta_a(x) := s\big(x \leftrightarrow a\big) \qquad \delta_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

The $\owedge$ behaviour

For any finite sequence t_j (e.g., $(1, 0, \frac{1}{2}, \dots)$),

$$\underset{j}{\ovee}\, t_j \;=\; \min\Big(1, \sum_j t_j\Big)$$

If at most one element in t_j is nonzero, call it t_m, then

$$\underset{j}{\ovee}\, t_j \;=\; t_m$$

Our selectors will be *crisp* (values in $\{0, 1\}$) and *mutually exclusive*. Their $1/2$-gated versions take values in $\{0, \frac{1}{2}\}$ and remain mutually exclusive. Hence $\ovee$ simply *picks the unique active value*.

Unary example: synthesizing $f : T \to T$

$$f(x) \;=\; \begin{cases} 1 & \text{if } x = 1 \\ 1 & \text{if } x = \frac{1}{2} \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

Selectors: $\delta_a(x) \in \{0, 1\}$ with $\delta_a(x) = 1$ iff $x = a$

Synthesis (gated by constants):

$$\varphi_f(x) \;:=\; \big(\delta_1(x) \owedge \mathbf{1}\big) \;\ovee\; \big(\delta_{1/2}(x) \owedge \mathbf{1}\big) \;\ovee\; \big(\delta_0(x) \owedge 1/2\big)$$

Simplification: Since $p \owedge \mathbf{1} = p$,

$$\varphi_f(x) \;\equiv\; \delta_1(x) \;\ovee\; \delta_{1/2}(x) \;\ovee\; \big(\delta_0(x) \owedge 1/2\big)$$

Tuple selectors

For a tuple $\mathbf{a} = (a_1, \ldots, a_n) \in T^n$ set

$$\Delta_{\mathbf{a}}(\mathbf{x}) \;:=\; \delta_{a_1}(x_1) \owedge \cdots \owedge \delta_{a_n}(x_n)$$

Because each $\delta_{a_i}(x_i) \in \{0, 1\}$:

- $\Delta_{\mathbf{a}}(\mathbf{x}) = 1$ exactly when $\mathbf{x} = \mathbf{a}$
- $\Delta_{\mathbf{a}}(\mathbf{x}) = 0$ otherwise.

Disjointness: If $\mathbf{a} \neq \mathbf{b}$, then $\Delta_{\mathbf{a}}(\mathbf{x}) \owedge \Delta_{\mathbf{b}}(\mathbf{x}) = 0$ for all $\mathbf{x}$. Thus the $\Delta_{\mathbf{a}}$ partition T^n into crisp, disjoint selectors.

Binary example

Define $g : T^2 \to T$ by the table

$g(x,y)$	$y=0$	$y=\frac{1}{2}$	$y=1$
$x=0$	0	0	0
$x=\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$x=1$	0	$\mathbf{1}$	$\mathbf{1}$

$$\varphi_g(x,y) = \Delta_{(1,1)}(x,y) \circledvee \Delta_{(1,\frac{1}{2})}(x,y)$$
$$\circledvee \left(1/2 \circledwedge \left(\Delta_{(\frac{1}{2},1)}(x,y) \circledvee \Delta_{(\frac{1}{2},\frac{1}{2})}(x,y)\right)\right)$$

Selectors are disjoint, so at any input exactly one of the four shown selectors is 1: the first block returns 1 on $(1,1)$ and $(1,\frac{1}{2})$; the gated block returns $\frac{1}{2}$ on $(\frac{1}{2},1)$ and $(\frac{1}{2},\frac{1}{2})$; elsewhere both blocks are 0.

Synthesis by cases (general construction)

Let $f : T^n \to T$. For $t \in T$ write c_t for the matching constant $(c_0 = \mathbf{0},\ c_{\frac{1}{2}} = 1/2,\ c_1 = \mathbf{1})$

$$\varphi_f(\mathbf{x}) := \bigovee_{\mathbf{a} \in T^n} (\Delta_{\mathbf{a}}(\mathbf{x}) \ \owedge \ c_{f(\mathbf{a})})$$

Because the $\Delta_{\mathbf{a}}$ are crisp and mutually exclusive, $\ovee$ picks the unique active gate, so for every input $\mathbf{x}$ we have $\varphi_f(\mathbf{x}) = f(\mathbf{x})$.